বিদ্যাসাগর বিশ্ববিদ্যালয়

VIDYASAGAR UNIVERSITY

Question Paper

B.Sc. Honours Examinations 2020

(Under CBCS Pattern)

Semester - III

Subject: MATHEAMATICS

Paper: GE3T

Full Marks : 60

Time : 3 Hours

Candiates are required to give their answer in their own words as far as practicable.
The figures in the margin indicate full marks.

DIFFERENTIAL EQUATION & VECTOR CALCULUS

Answer any ***three*** from the following questions : 3×20

1. (i) State Picard's theorem of successive approximations.

   Use Picard's method to compute approximately the value of y when $x = 0.1$ from the initial value problem $\frac{dy}{dx} = x + y$ where $y(0) = 1$

   (ii) Define regular singular point and irregular singular point of a differential equation $p_0(x)\frac{d^2y}{dx^2} + p_1(x)\frac{dy}{dx} + p_{20}(x)y = 0$

(iii) Solve : $\frac{dx}{xy}=\frac{dy}{y^2}=\frac{dz}{xyz-2x^2}$.

(iv) Show that $[a+\beta, \beta+\gamma, \gamma+a]=2[a\ \beta\ \gamma]$ 3+6+3+5+3

2. (i) Show that $(yz+xyz)dx+(zx+xyz)dy+(xy+xyz)dz=0$ is integrable and hence solve it.

(ii) Find the particular integral of $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=x^2e^{3x}$.

(iii) Find the value of a for which the vector $V=(x+3y)i+(y+az)j+(x+az)k$ is solenoidal.

(iv) If $\sum_{m=0}^{\infty} C_m x^{r+m}$ is assumed to be a solution of $x^2y''-xy'-3(1+x^2)y=0$ then find the values of r.

(v) Find the Wronskian of the following set $\{\sin 3x, \cos 3x\}$. 2+5+3+4+3+3

3. (i) Solve the differential equation

$\frac{d^2y}{dx^2}+9y=\frac{1}{4}\operatorname{cosec} 3x$ by the method of variation of parameters.

(ii) Find the equlilbrium point of the system of differential equations

$x=e^{x-1}-1$ and $\dot{y}=ye^x$.

(iii) If $V=xyi-z^2j+xyzk$, then show that $\int_c V.dr=\frac{1}{3}$ when the integral is taken from (0,00) to (1,1,1) along curve $r=ti+t^2j+t^3k$..

(iv) Let $y=\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ be a solution of the system of equation $\begin{bmatrix} y_1' \\ y_2' \end{bmatrix}=\begin{bmatrix} 0 & 1 \\ a & b \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ where $a, b \in R$. Then prove that every solution $y(x)\to 0$ as $x\to\infty$ if $a<0$ and $b<0$.

8+3+5+4

4. (i) Solve the differential equaiton :

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} = x + e^x \sin x$$ by the method of undetermined coefficient.

(ii) Solve :

$$\frac{dx}{dt} + \frac{dy}{dt} + 2x + y = 0$$

$$\frac{dy}{dt} + 5x + 3y = 0$$

(iii) If $r = (a\cos t)i + (a\sin t)j + (at\tan\alpha)k$, then show that $\left|\frac{dr}{dt} \times \frac{d^2r}{dt^2}\right| = a^2 \sec\alpha$.

(iv) Show that $x = 0$ is the regular singular point of the differential equation

$$2x^2\frac{d^2y}{dx^2} 7x(x+1)\frac{dy}{dx} - 3y = 0.$$

7+7+4+2

5. (i) Find the power series solution of the equaiton $(x^2+1)y'' + xy' - xy = 0$ about $x = 0$.

(ii) Show that $\frac{dy}{dx} = 3y^{\frac{2}{3}}$, $y(0) = 0$ has more than one solution and indicate the possible region.

(iii) If $a = (3,1,1)$, $b = (1,-2,2)$, $c = (4,1,3)$, than calculate $a.(b \times c)$ and state geometrical meaning of the result.

(iv) Solve the equation $\frac{dx}{dt} = -wy$ and $\frac{dy}{dt} = wx$ and show that point (x, y) lies on a circle.

8+3+5+4

6. (i) Find the value of the constant d such that the vectors $(2i - j + k)$, $(i + 2j - 3k)$ and $(3i + dj + 5k)$ are coplanar.

(ii) If u and v be two independent solutions of the linear equation

$$\frac{d^2y}{dx^2}+p\frac{dy}{dx}+Qy=0$$

then prove that the Wronskian $W(u,v)$ is given by $W(u,v)=Ae^{-\int pdx}$ where A is constant.

(iii) Solve $\frac{d^4y}{dx^4}+a^4y=0$.

(iv) Solve $x^2\frac{d^2y}{dx^2}-x(x+2)\frac{dy}{dx}+(x+2)y=0$ given $y=x$ is a solution.

(v) Find the order and degree of the differential equation $y=x\frac{dy}{dx}+a\frac{dx}{dy}$. 4+5+4+5+2

GROUP THEORY - I

Answer any ***three*** from the following questions : 3×20

1. (a) Prove that the see of all ratinal numbers, other than 1, forms a commutative group with respect to the composition * defined by $a*b=a+b-1$ for $a,b\in\mathbb{Q}-\{1\}$. 7

   (b) Show that intersection of two subgroup of a group G is a subgroup of G. Is the result true fo union ? Justify. 7

   (c) Prove that any two left cosets of a group G have the same number of elements. 6

2. (a) Prove that if $a^2=e$, for all $a\in G$, then G is an abelian group. 5

   (b) Find all cyclic subroups of the symmetric groups S_3. 5

   (c) Prove that every subgroup of a cyclic group is cyclic. 5

   (d) Let G be a group and H be a subgroup of G. Let $g\in G$ be fixed. Prove that the subset $K=\{ghg^{-1}:h\in H\}$ forms a subgroup of G. 5

3. (a) Prove that the alternating group A_3 is a normal subgroup of the symmetric group S_3. 5

(b) Prove that every group of order less than 6 is commutative. 6

(c) State and prove Lagrange's theorem on finite group. 9

4. (a) Prove that every proper subgroup of a group order 6 is cyclic. 7

(b) Prove that every group of prime order is cyclic. Is the converse true ? Justify. 8

(c) Let H be a subgroup of a group G and $[G:H]=2$. Prove that H is a normal subgroup of G. 5

5. (a) Let $G=(\mathbb{Z},+)$ and a mapping $\varphi:G\to G$ be defined by $\varphi(x)=x+1,\ x\in G$. Examine if φ is a homomorphism. 3

(b) If H be a subgroup of a commutative group G, prove that the quotient group $\frac{G}{H}$ is commutative. 3

(c) Prove that the center $Z(G)$ of a group G is a normal subgroup of G. 6

(d) State and prove the first isomorphism theorem. 8

6. (a) Let $\varphi:(G,0)\to(G_1,*)$ be a homomorphism. Then prove that ker φ is a normal subgroup of G. 7

(b) Show that the group $(\mathbb{Q},+)$ and $(\mathbb{R},+)$ are not isomorphic. 6

(c) Let $\varphi:(G,0)\to(G_1,*)$ be an isomorphism. Then prove that G_1 is cyclic if and only if G is cyclic. 7.

THEORY OF REAL FUNCTIONS AND INTRODUCTION TO METRIC

Answer any ***three*** from the following questions : 3×20

1. (a) State and prove Rollle's theorem. 8

(b) In the Mean value theorem $f(h)=f(0)+hf(\theta h),\ 0<\theta<1$, show that the limiting value of θ as $h\to 0^+$ is $1/2$ or $1/\sqrt{3}$ according as $f(x)$ is $\cos x$ or $\sin x$. 8

(c) Write the geometrical interpretation of Lagrange's Mean Value theorem. 4

2. (a) State and prove Darboux's theorem. 6

(b) Prove that $\cos x > x - \frac{x^2}{2}$, if $0<x<\frac{\pi}{2}$. 6

(c) Expand the function $(1+x)^p$ in power of x in infinite series stating in each case the conditions under which the expansion is valid. 8

3. (a) State and porve Maclaurin's theorem with Cauchy's form of reamainder. 8

(b) Show that the maximum value of $x+\frac{1}{x}$ is less than its minimum value. Explain why ? 6

(c) Show that the largest rectangle with a given perimeter is a square. 6

4. (a) Is the function

$$f(x)=\sin\left(1/x\right),\quad x\neq 0$$
$$=0\qquad x=0$$

continuous at the origin ? 6

(b) Show that the functin f defined by $f(x)=\sin x, x\in\mathbb{R}$ is uniformly continuous on $\mathbb{R}$.

(c) If $-1<x<1$, then prove $\lim_{n\to\infty} x^n = 0$ where n is a positive integer. 8

5. (a) Define metric space. Show that the Euclidean space $\mathbb{R}^n$ is a metric space. 8

(b) Let $C[a,b]$ be the collection of all real valued continuous functions over the closed interval $[a,b]$. Let $d : C[a,b]\times C[a,b] \to \mathbb{R}$ be defined by

$$\sup_{a\le t\le b}\left\{\left|f(t)-g(t)\right|, \forall f, g\in[a,b]\right\}.$$

(c) Show that $\forall x, y\in\mathbb{R}, d(x,y)=\left|\tan^{-1}x-\tan^{-1}y\right|$ is a metric on $\mathbb{R}$ and also bouded. 6

6. (a) Let $A=\left\{(x,y):x^2+y^2=1\right\}$, $B=\left\{(x,y):(x-1)^2+y^2=1\right\}$. Find the diameter of $A\cup B$ and $A\cap B$ with respect to usual metric. 6

(b) Define open spheres and closed spheres. Prove that, in a metric space, any open sphere is an open set. 8

(c) Prove that arbitrary intersection of a closed set in a metric space is a closed set. 6