বিদ্যাসাগর বিশ্ববিদ্যালয়

VIDYASAGAR UNIVERSITY

Question Paper

B.Sc. Honours Examinations 2020

(Under CBCS Pattern)

Semester - III

Subject: MATHEAMATICS

Paper: C5T

(Theory of Real Functions and Introduction to Metric Space)

Full Marks : 60

Time : 3 Hours

Candiates are required to give their answer in their own words as far as practicable.

The figures in the margin indicate full marks.

Answer any ***three*** from the following questions : 3×20

1. (a) Examine with reason whether $\lim_{x\to 0}\left(\sin\frac{1}{x}+x\sin\frac{1}{x}\right)$ exist or not. 2

(b) Give examples of a function which is 5

(i) Continues and bounded on $\mathbb{R}$, attains its suprimum but not infimum.

(ii) Continues and bounded on $\mathbb{R}$, attains its infimum but not its suprimum.

(iii) Continunes and bounded on an interva, but attains neither its suprimum nor infimum.

(c) Let $[a, b]$ be a closed and bounded interval and $f:[a,b]\to\mathbb{R}$ be continuous on $[a, b]$. If $f(a)$ and $f(b)$ are opposite sign then show that there exists at least a point c in the open interval (a, b) such that $f(c) = 0$. 6

(d) Does Rolle's theorem hold for $f(x)=1-|1-x|$ in $[0, 2]$ Justify. 2

(e) Let $f(x)=\begin{cases} x\left\{1+\frac{1}{3}\left(\log x^2\right), & x\neq 0 \\ 0 & , \quad x=0\end{cases}$ Show that f is continuous at $x = 0$ but not derivable there. 5

2. (a) Show that the Dirichlet's function is everywhere discountnuous on $\mathbb{R}$. 2

(b) Let $[a, b]$ be a closed and bouded interval and a function $\mathbb{R}$ be continuous on $[a, b]$. If $f(a)\neq f(b)$ then f attains every value between $f(a)$ and $f(b)$ at least once in (a, b). Is the converse ture ? Justify. 5

(c) Give and example of a function f defined on an interval I such that 4

(i) f has jump discontinuity at a point of I.

(ii) f has removable discontinuity at a point of I.

(iii) f has infinite discontinuity at a point of I.4.

(d) (i) Prove that for no real value of k, the equation $x^3-12x+k=0$ has two real roots in $[-1,.1]$. 2+3

(ii) Prove that there does not exist a function φ such that $\varphi^{-}(x)=f(x)$ on $[0, 2]$ where $f(x)=x-[x]$.

(e) Prove that $x-\frac{x^3}{x}<\sin x<-\frac{x^3}{6}+\frac{x^5}{120}$ for all $x>0$. 4

3. (a) Prove that there exists $x \in \left(0, \frac{\pi}{2}\right)$ such that $x = \cos x$. 2

(b) Let $D \subset \mathbb{R}$ and a function $f : D \to \mathbb{R}$ be uniformly continuous on *D*. If $\{x_n\}$ be a Cauchy sequence in *D* then show that $\{f(x_n)\}$ is a Cauchy sequence in $\mathbb{R}$. If we drop the condition "uniformity", then is the above reseult hold ? Justify. 5

(c) If *f*(*x*) be differentiable at $x = a$ show that 2

$$\lim_{x \to a} \frac{(x+a) f(x) - 2af(a)}{x-a} = f(a) + 2af'(a).$$

(d) (i) State Rolle's theorem. Is the set of conditions of Rolle's theorem a necessary condition ? Justify.

(ii) If a function *f* is continuous at a point $x = 0$, prove t hat $xf(x)$ is derivable at $x = 0$. 5

(e) State and prove Lagrange's mean value theorem. Give its geometrical signifincance. 6

4. (a) $f : [0,1] \to \mathbb{R}$ is continuous on [0,1] and *f* assumes only rational values. If $f\left(\frac{1}{2}\right) = \frac{1}{2}$, prove that $f(x) = \frac{1}{2}$ for all $x \in [0,1]$. 2

(b) (i) Give an examples to show that a function which is continuous on an open bouded interval may not be uniformly continuous there. 2+3

(ii) Let f be continuous on [a, b] and $f(x) = 0$ when *x* is rational. Show that $f(x) = 0$ for every $x \in [a,b]$.

(c) Find $f'(0)$ [if exist] for the function $f(x) = \begin{cases} 3 + 2x, & -\frac{3}{2} < x \le 0 \\ 3 - 2x, & 0 < x < \frac{3}{2} \end{cases}$ 2

(d) Prove that between any two real roots of $e^x \sin x = 1$, there exist at least one real root of $e^x \cos x + 1 = 0 \cdot$ 2

(e) Expand $\sin x,\ x \in \mathbb{R}$, in powers of x by Tailor's series expansion. 3

(f) Find the minimum value (if exist) of the function defined by $f(x) = x^x, (x > 0)$ 3

(g) Show that the greatest value of $x^m y^n, (x > 0, y > 0)$ and $x + y = k$ (k = constant) is $\dfrac{m^n n^n k^{m+n}}{(m+n)^n}$. 3

5. (a) Prove that between any two real roots of $e^x \sin x = 1$ there exist at least one real root of $e^x \cos x + 1 = 0 \cdot$ 4

(b) A function f is thrice differentiable on $[a,b]$ and $f(a) = f(b) = 0$ and also $f'(a) = f'(b) = 0$. Prove that the second derivative of f vanishes at c, where $a < c < b$. 4

(c) Define discrete and pseudo metric space. 2

(d) On the real line $\mathbb{R}$, show that a singleton set is not an open set. 2

(e) Let X be the set of all sequences of real numbers containing only a finite number of non-zero element. Let $d : X \times X \to X$ be defined by $d(\{x_n\}, \{y_n\}) = \left\{\sum_{r=1}^{\infty} (x_r - y_r)^2\right\}^{\frac{1}{2}}$. 6

(f) Give an example to show that the continuous image of an open bounded interval may not be an open bouded interval. 2

6. (a) In the mean value theorem $f(x+h) = f(x) + hf'(x+\theta h), 0 < \theta < 1$, prove that $\lim_{h \to 0+} \theta = \dfrac{1}{2}$ if $f(x) = \sin x$. 4

(b) Show that any discrete metric space is a complete metric space. 2

(c) Show that in any metric space, a finite set has no limit point. 2

(d) Show by example that in any metric space, the Cantor intersection theorem may not hold good if any of the following conditions is not satisfied : 8

(i) $\{F_n\}$ is a sequence of closed sets.

(ii) $\delta(F_n) \to 0$ as $n \to \infty$ where $\delta(A)$ denotes the diameter of the set A.

(e) We know in a metric space (X, d), "the union of a finite number of closed sets is closed". In this result if we drop the finiteness, then is the result hold good ? Justify. 4