Individual Risk Models

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This is continuation of notes given last class before corona holiday.

For an insurance organization , let the loss of a segment of its risks is denoted by S. So, if company issued n independent policies, the *individual risk model* defines S as

$$S = X_1 + X_2 + \ldots + X_n$$

where X_i is the loss on i^{th} policy or i^{th} insured unit. In individual risk model it is assumed that n is fixed and known. The number of times claim will be made within the insured time is either 0 or 1, so some X_i in S may be zero. This drawback can be overcame by using *collective risk model* which will be discussed later. The main objective in modellling claim sizes as well as frequency of claims is to decide how premiums can be changed to avoid the ruin of the insurance company.

$X_1, X_2, \ldots, X_n$ are assumed to be independent. In the individual risk model, each component X_i in S is expressible as

$$X_i = I_i B_i$$

for $i = 1, 2, \ldots, n$ where

$$I_i = \begin{cases} 1 & \text{if } \; i^{th} \quad \text{claim is made} \\ 0 & \quad \text{otherwise} \end{cases}$$

We assume $P[I_i = 1] = q_i = 1 - P[I_i = 0]$ for $i = 1, 2, \ldots, n$. Then X_i is a compound Bernoulli random variable given as

$$X_i = \begin{cases} B_i & \text{with probability} \quad q_i \\ 0 & \text{with probability} \quad 1 - q_i \end{cases}$$

for $i = 1, 2, \ldots, n$.

Let us take a eaxmple of automobile insurance and try to obtain the $E(X)$ and $Var(X)$ for single insurance policy. Then we will obatin or approximate $E(S)$ and $Var(S)$.

Suppose an autimobile insurance providing collison coverage above a 250 deductible upto maximum claim of 2000. We assume that the probability that the claim will within the insured period is 0.15.

Now, $X = IB$ with $P(I = 0) = 0.85$ and $P(I = 1) = 0.15$. B is the amount of claim. The conditional distribution of B given $I = 1$ is modelled by mixed distribution such that $P(B = 2000|I = 1) = 0.1$ and conditional pdf of B given $I = 1$ when $0 < x < 2000$ is $f_{B|I=1}(x) = c\left(1 - \frac{x}{2000}\right)$. Then the conditional c.d.f. of B given $I = 1$ is given by

$$F_B(B \leq x|I = 1) = \begin{cases} 0 \quad \text{if} \quad x < 0 \\ 0.9\left[1 - \left(1 - \frac{x}{2000}\right)^2\right] \quad \text{if} \quad 0 \leq x < 2000 \\ 1 \quad \text{if} \quad x \geq 2000 \end{cases}$$

So, the distribution function of X is

$$\begin{aligned} P(X \leq x) &= P(IB \leq x) \\ &= P(IB \leq x|I = 1)P(I = 1) + P(IB \leq x|I = 0)P(I = 0) \end{aligned}$$

So, marginal cdf of X is

$$F_X(x) = \begin{cases} 0.(0.15) + 0.(0.85) = 0 \quad \text{if} \quad x < 0 \\ 0.9\left[1 - \left(1 - \frac{x}{2000}\right)^2\right].0.15 + 1.(0.85) \quad \text{if} \quad 0 \leq x < 2000 \\ 1.(0.15) + 1.(0.85) = 1 \quad \text{if} \quad x \geq 2000 \end{cases}$$

From here we can obtain the combination of p.m.f. and p.d.f. of X and obtain $E(X)$ and $Var(X)$.

But we can obtain these more directly. Now.

$$E(X) = E(E(X|I))$$

and

$$Var(X) = Var(E(X|I)) + E(Var(X|I))$$

Let us write

$$\mu = E(B|I = 1)$$

and

$$\sigma^2 = Var(B|I = 1).$$

$E(X|I = 1) = E(IB|I = 1) = E(B|I = 1) = \mu$ and $E(X|I = 0) = 0$.

So,

$E(X) = E(X|I = 1)\cdot P(I = 1)+E(X|I = 0)\cdot P(I = 0) = \mu P(I = 1) = \mu q(\text{say})$

Now, $Var(X|I = 1) = Var(B|I = 1) = \sigma^2$ and $Var(X|I = 0) = 0$

Hence, $E(Var(X|I)) = \sigma^2 \cdot P(I = 1) + 0 \cdot P(I = 0) = \sigma^2 \cdot q$

Also, $Var(E(X|I)) = E(E(X|I) - E(X))^2 = E(E(X|I))^2 - (E(X))^2 = \mu^2 \cdot P(I = 1) - (\mu \cdot q)^2 = \mu^2 \cdot q \cdot (1 - q)$

Hence,

$$Var(X) = \mu^2 \cdot q \cdot (1 - q) + \sigma^2 \cdot q$$

In our automobile insurance problem,

$$\begin{aligned} \mu &= 2000P(B = 2000|I = 1) + \int_0^{2000} 0.9\frac{x}{1000}\left(1 - \frac{x}{2000}\right) dx \\ &= 2000 \cdot 0.1 + 600 = 800 \end{aligned}$$

Again

$$\begin{aligned} E(B^2|I=1) &= (2000)^2 \cdot 0.1 + \int_0^{2000} 0.9 \frac{x^2}{1000}\left(1 - \frac{x}{2000}\right) dx \\ &= 400000 + 600000 = 1000000 \end{aligned}$$

So,

$$\sigma^2 = E(B^2|I=1) - (E(B|I=1))^2 = 1000000 - 800^2 = 360,000.$$

Hence,

$$Var(X) = \mu^2 \cdot q \cdot (1-q) + \sigma^2 \cdot q = 800^2 \cdot 0.15 \cdot (1 - .15) + 360000 \cdot 0.15 = 135600$$

So, in individual risk model, if X_i is the loss for i^{th} insured unit and B_i is the amount of claim and q_i is probability that claim will be made then $X_i = I_i B_i$ as before and

$$E(X_i) = \mu_i q_i \qquad \text{and} \quad Var(X_i) = \mu_i^2 q_i (1 - q_i) + \sigma_i^2 q_i$$

where $\mu_i = E(B_i|I_i = 1)$ and $\sigma_i = Var(B_i|I_i = 1)$

The distribution of $S = \sum_{i=1}^{n} X_i$, the total loss for a portfolio of n independent policies, can be approximated as normal for large n with approximate mean $E(S) = \sum_{i=1}^{n} E(X_i)$ and approximate variance $Var(S) = \sum_{i=1}^{n} Var(X_i)$.

<u>Exercise 1:</u>The policyholders in an automobile company fall into two classes

Class k	Number in class n_k	Claim Probability q_k	Distribition of claim amount Parameters of Truncated exponential λ	 L
1	500	0.10	1	2.5
2	2000	0.05	2	5

The truncated exponential distribution is defined by the df

$$F(x) = \begin{cases} 0 \quad \text{if} \quad x < 0 \\ 1 - e^{-\lambda x} \quad \text{if} \quad 0 \le x < L \\ 1 \quad \text{if} \quad x \ge L \end{cases}$$

This is mixed distribution with p.d.f. $f(x) = \lambda e^{-\lambda x}$ if $0 < x < L$ and mass $e^{-\lambda L}$ at L. Again the probability that total claim exceed the amount collected from policy holders is 0.05. Moreover, it wants each individual's share of this amount to be proportional to individual's expected claim. The share of the individual j with mean $E(X_j)$ would be $(1+\theta)E(X_j)$. We assume that relative security loading θ is same for two classes. Calculate θ.

(Hint: $S = \sum_{i=1}^{500} X_{1i} + \sum_{i=1}^{2000} X_{2i}$, X_{ji}, $j = 1, 2$ *are modelled by truncated exponential. Obtain* $E(S)$ *and* $Var(S)$ *and calculate* θ *such that* $P(S \ge (1+\theta)E(S)) = 0.05$*)*

Exercise 2 The portfolio of an insuarance company consists of 350 policies. For each policy the probability q of a claim is 0.002 and the benefit amount, given that there is a claim, has gamma distribution with scale parameter 0.25 and shape parameter 50. Suppose S denotes the total claims for portfolio for a short period. Find $P(S > 71000)$, $P(S < 69,000)$ and $P(69,000 < S < 71,000)$. What relative security loading , θ, should be used so the company can collect an amount of 99th percentile of the distribution of total claims.

Exercise 3 : The following table shows the data for a life insurance company which which issues one year term life contracts

Group	Death Rate	Benefit Amount	Number of lives
1	0.001	1	100
2	0.002	2	300
3	0.003	3	400
4	0.003	4	350

Find the expectation and variance of the total risk from the portfolio to the company. The company wants to collect, from these individuals, an amount equal to the 95^{th} percentile of the distribution of total claims. The share for individual j with mean $E(X_j)$ would be $(1+\theta)E(X_j)$. Calculate the relative security loading θ.

Reinsurance Reinsurance is an agreement between two insurance companies, one known as insurer and the other as reinsurer. Under this agreement, claims that occur in a fixed period of time are split between the insurer and the reinsurer in a specific proportion. Thus the insurer is effectively insuraing a part of risk with reinsurer and pays the premium to the reinsurer for this cover.
Consider following
A life insurance company covers $16,000$ lives for 1 year term life insurance in amounts shown below

Benefit Amount	Number Covered
10,000	8,000
20,000	3,500
30,000	2,500
50,000	1,500
1,00,000	500

The probability of a claim q for each of $16,000$ lives,assumed to be independent, is 0.02. The company wants to set a *retention limit* . For each life the retention limit is the amount below which this company will retain the insurance and above which it will purchase reinsurance coverage from a reinsuring company.

Suppose the retention limit is $20,000$. The company will retain upto $20,000$ on each life and purchase reinsurance for the difference between the benefit amount and $20,000$ for each individual in a group with benefit amounts in excess of $20,000$. As a decision criterion, the company wants to minimize the probability that retained claims plus the amount it pays for reinsurance will exceed 8,250,000. Reinsurance is available at a cost of 0.025 per unit of coverage (that is , 125% of the expected claim amount per unit).

We will consider the block of business is closed and new plocies sold during the year are not to enter this decision process.

Let us first calculate the probability that company's retained claims plus cost of reinsurance exceeds 8,250,000.

We will do all the calculation at the unit of 10,000. let S be the retained claims paid when the retention limit is 2(20,000). Our portfolio of the retained business is given by

	Retained Amount	Number Covered
k	b_k	n_k
1	1	8,000
2	2	8,000

and $E(S) = \sum_{k=1}^{2} n_k b_k q_k = 8000 \cdot 1 \cdot 0.02 + 8000 \cdot 2 \cdot 0.02 = 480$

and $Var(S) = \sum_{k=1}^{2} n_k b_k^2 q_k (1-q_k) = 8000 \cdot 1 \cdot 0.02 \cdot 0.98 + 8000 \cdot 4 \cdot 0.02 \cdot 0.98 = 784$

In addition to reatined claims,S, there is a cost of reinsurance premiums. The total coverage of the plan is

$$8000(1) + 3500(2) + 2500(3) + 1500(5) + 500(10) = 35000$$

the retained coverage for the plan is $8000(1) + 8000(2) = 24000$.
So, the total amount reinsured is $35000 - 24000 = 11000$ and reinsurance cost is $11000 \cdot 0.025 = 275$. So, retained cost plus reinsurance cost at retention limit 2 is $S + 275$, hence the the probability that company's retained claims

plus cost of reinsurance exceeds $8,250,000$ is

$$\begin{aligned} P(S+275>825) &= P(S>550) \\ &= P\left(\frac{S-E(S)}{\sqrt{Var(S)}} > \frac{550-E(S)}{\sqrt{Var(S)}}\right) \\ &= 0.0062 \end{aligned}$$

The distribution of S is approximated by Normal distribution.

Now, suppose our problem is obtain the retention limit such that it minimizes the above probability (probability that company's retained claims plus cost of reinsurance exceeds $8,250,000$). It is given that the limit is between $30,000$ to $50,000$.

Let l be the retention limit. The following table gives the retention limit and number covered

k	Retained Amount b_k	Number Covered n_k
1	1	8,000
2	2	3,500
3	3	2,500
4	l	2,000

Let S be the retained claims paid,

$$E(S) = \sum_{k=1}^{4} n_k b_k q_k = 8000\cdot1\cdot0.02+3500\cdot2\cdot0.02+2500\cdot3\cdot0.02+2000\cdot l\cdot0.02 = 450+40l$$

and

$$\begin{aligned} Var(S) &= \sum_{k=1}^{4} n_k b_k^2 q_k(1-q_k) \\ &= 8000\cdot1\cdot0.02\cdot0.98+3500\cdot4\cdot0.02\cdot0.98+2500\cdot9\cdot0.02\cdot0.98+2000\cdot l^2\cdot0.02\cdot0.98 \\ &= 872.2+39.2l^2 \end{aligned}$$

As before, The total coverage of the plan is

$$8000(1) + 3500(2) + 2500(3) + 1500(5) + 500(10) = 35000$$

but the retained coverage for the plan is $8000(1)+3500(2)+2500(3)+2000(l) = 22500 + 2000l$.

Therefore, the amount reinsured is $35000 - 22500 - 2000l = 12500 - 2000l$ and cost of reinsurance is $(12500 - 2000l) \cdot 0.025 = 312.5 - 50l$. So, retained cost plus reinsurance cost is $S + 312.5 - 50l$, so the probability that it exceeds 825 is

$$\begin{aligned} P(S + 312.5 - 50l > 825) &= P(S > 512.5 + 50l) \\ &= P\left(\frac{S - E(S)}{\sqrt{Var(S)}} > \frac{512.5 + 50l - E(S)}{\sqrt{Var(S)}}\right) \\ &= P\left(\tau > \frac{512.5 + 50l - 450 - 40l}{\sqrt{872.2 + 39.2l^2}}\right) \\ &= 1 - \Phi\left(\frac{62.5 + 10l}{\sqrt{872.2 + 39.2l^2}}\right) \end{aligned}$$

Miminization of above probability is equivalent to maximization of $\Phi\left(\frac{62.5+10l}{\sqrt{872.2+39.2l^2}}\right)$. As, $\Phi(.)$ is a increasing function $\Phi\left(\frac{62.5+10l}{\sqrt{872.2+39.2l^2}}\right)$ will be maximized when $f(l) = \frac{62.5+10l}{\sqrt{872.2+39.2l^2}}$ is maximized. Now, from equation

$$f'(l) = 0$$

we get $l = 3.578256$. Hence the retention limit will be $l \cdot 10,000 = 35782.56$.

Exercise 4: A life insurance company covers $10,000$ lives for 1 year term life insurance in amounts shown below

Benefit Amount	Number Covered
10,000	5,000
25,000	2,500
35,000	1,500
65,000	500
80,000	500

Suppose the probability q of a claim is 0.01 for each of these $10,000$ policies. Reinsurance is available at a cost of 0.02 per unit of coverage. Calculate the retention limit that minimizes the probability that company's retained claims paid plus cost of reinsurance exceeds $4,250,000$. Assume that the limit is between 25000 and 35000.

Also, for same data, calculate the retention limit that minimizes the probability of the total cost exceeding $44,50,000$. Assume that the limit is between $20,000$ and $30,000$.

Send the solutions of the exercises at *tuhinsubhra.bh@gmail.com*. **Do the exercises on your workbook clearly and scan those to send me in may mail** Notes and exercises on collective risk model will be sent on next week.