MATHEMATICAL MODELLING

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What is Mathematical Modeling ?

A **mathematical model** is a description of a system using Mathematical concepts and language.
The process of developing a mathematical model is termed **mathematical modeling**.

Mathematical models are used in :

- Natural sciences (such as physics, biology, earth sciences, meteorology etc.)
- Engineering disciplines (such as computer Sciences, artificial intelligence etc.)
- Social Sciences (such as economics, psychology, sociology, etc.)

Elements of a mathematical model

Mathematical models can take the forms, like :

- Dynamical Systems,
- Statistical Models,
- Differential Equations,
- Game theoretic Modeling.

Ecosystem model:

- Ecosystem models or ecological models are mathematical representations of ecosystems. Typically they simplify complex food-webs down to their major components or trophic levels and quantify these as either numbers of organisms, biomass or the inventory/concentration of some pertinent chemical element (for instance, carbon or a species' nutrient such as nitrogen or phosphorus).

Complexity of Ecosystem model:

- Ecosystem models are a development of theoretical ecology that aim to characterize the major dynamics of ecosystems, both to synthesize the understanding of such systems and to allow predictions of their behavior.

- Because of the complexity of ecosystems (in terms of numbers of species/ecological interactions), ecosystem models typically simplify the systems which are studied to a limited number of pragmatic components.

Modeling factors:

- **Ignorance:** While understood in broad outline, the details of a particular food- web may not be known; this applies both to identifying relevant species, and to the functional responses linking them (which are often extremely difficult to quantify)
- **Computation:** Practical constraints on simulating large numbers of ecological elements; this is particularly true when ecosystem models are embedded within other spatially-resolved models (such as physical models of terrain or ocean bodies, or idealized models such as cellular automata or coupled map lattices).
- **Understanding:** Depending upon the nature of the study, complexity can con- found the analysis of an ecosystem model; the more interacting components a model has, the less straightforward it is to extract and separate causes and con- sequences; this is compounded when uncertainty about components obscures the accuracy of a simulation.

Structure of an Ecosystem model:

- The simplification process of this kind of model typically reduces an ecosystem to a small number of state variables.
- Depending upon the system under study, these may represent ecological components in terms of number of discrete individuals or quantify the component more continuously as a measure of the total biomass.
- The components are then linked together by mathematical functions that describe the nature of the relationships between them.

Basic Concepts and Terminologies:

- **Growth Rate:**
- As we know that the population changes over time, so it is important to know that how it is changing or more precisely what is its time rate of change which we call the growth rate.
- In short The growth rate of a population is the rate of change of its density or size per unit time. It is determined by the difference of birth rate and the death rate.

- **Birth Rate:**
- The birth rate of a population is the maximum production of new individuals per unit time under certain ideal conditions (i.e., without any ecological limiting factors, production being limited by physiological factors only).
- **Death Rate:**
- Death rate may be expressed as the number of individuals dying per unit time.
- **Half-saturation constants:**
- The concentration supporting an uptake rate one-half the maximum rate. i.e. the concentration supporting half of the maximum uptake rate.

Continuous Single Species Population Models:

- **Continuous Growth Model:** Single-species models are of relevance to laboratory studies in particular but, in the real world, can reflect a telescoping of effects which influence the population dynamics. Let $x_1\,(t)$ be the population of the species at time t, then the rate of change.
- $\frac{dx_1}{dt} = births - deaths + migration.$
- **Delay Model:** Most of the population model did not considered maturity time, and gestation period. But in reality these factors have an important role in population dynamics. Time delay is the time to reach maturity and the finite gestation period that is considered in the population dynamics and the corresponding differential equation is known as delay differential equation which is of the form
- $\frac{dN}{dt} = f(N(t), N(t-T))$

- **Deterministic Model:** A deterministic model is one in which every set of variable states is uniquely determined by parameters in the model and by sets of previous states of these variables. Therefore, deterministic models perform the same way for a given set of initial conditions.

- **Stochastic Model:** Environmental fluctuation is an important component of an ecosystem. Deterministic models in ecology do not usually incorporate environmental fluctuation. A stochastic model provides a more realistic picture of a natural system than its deterministic part as the environment is always subjected to random fluctuations that affect the system of population dynamics.

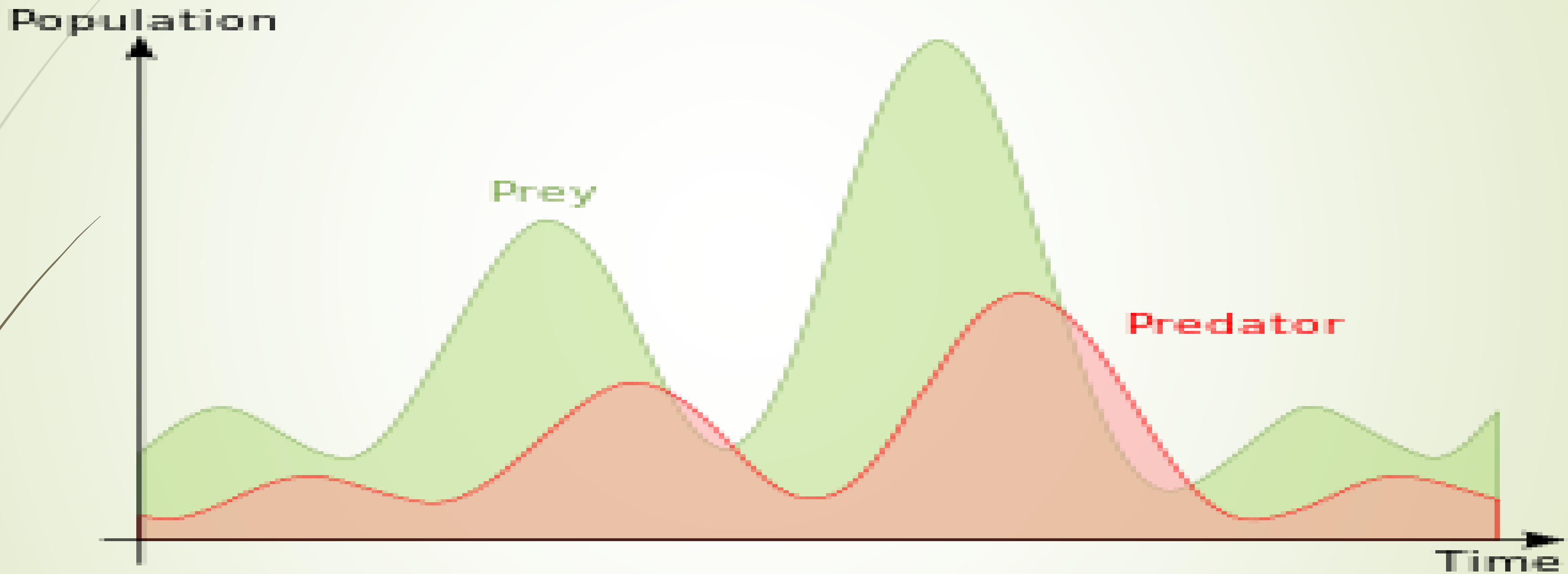
- **Fuzzy Model:** An ecological model is said to be Fuzzy model if one or more parameters of the model are fuzzy in nature.

Lotka-Volterra Model:

- The **Lotka–Volterra equations**, also known as the **predator–prey equations**, are a pair of first-order, nonlinear, differential equation frequently used to describe the dynamics of biological systems in which two species interact, one as a predator and the other as prey. The populations change through time according to the pair of equations:

- $\frac{dx}{dt} = \alpha x - \beta xy$
- $\frac{dy}{dt} = \delta xy - \gamma y$

- x is the number of prey (for example, rabbits);
- y is the number of some predator (for example, foxes);
- $\dot{x}$ and $\dot{y}$ represent the growth rates of the two populations over time;
- $\boldsymbol{t}$ represents time.
- $\alpha, \beta, \gamma, \delta$ are positive real parameters describing the interaction of the two species


Population
Prey
Predator
Time

Harvesting Model :

- In the population dynamics, especially in fishery, resource harvesting is very important part to make the population model more realistic. The harvesting in population biology has taken a definite shape by the work of Clark. For a single species population model, the mathematical form of the model is like:
- $\frac{dx}{dt} = F\,(x) - h(t)$
- where $F(x)$ takes different forms for logistic and Gompertz types of growth laws and $h(t)$ is removal rate or harvesting rate. For constant rate of harvesting $h(t)$ is constant and for other cases $h(t)$ has different forms.

- Again for multi (two)- species model, the system with harvesting is of the form:

- $dx/dt = F_1(x) - \alpha g_1(x, y) - h_1(t)$
- $dy/dt = F_2(y) + \beta g_2(x, y) - h_2(t)$

Discrete Age-structured Model :

We have developed the model with discrete-age scale and discrete-time. Hence the birth and death rates of different age groups are constant within age group and the changes in population in different stages i.e. the population at time (t+1) can be determined from the known population at time t. The present model is also one-sex model i.e. all changes are assumed to occur in female populations only and the male populations conform with these changes. Let the female population be divided into n age groups and the populations of all age groups at time t is $x_1(t)$, $x_2(t)$, $x_3(t)$, …….., $x_n(t)$, i.e. i-th suffix denotes i-th age group. We also assume the model to be deterministic and linear i.e. there is no random parameter/ variable in the model element and all changes are proportional to the population size.

Let $f_i, (i=1,2,\ldots,n; f_i \geq 0)$ represents the average number of female off-spring alive at time (t+1),born in the time interval (t, t+1) to each female who was in the age group (i-1, i) at time t.Now the number of females in the age group (i-1, i) at time t is $x_i(t)$,each of whom gives birth on an average to a certain number of number of female off-spring in the interval (t, t+1),of whom f_i remains alive at time $(t+1)$. So that the number of female off-spring in the age group (0,1), alive at time $(t+1)$,born out of these $x_i(t)$ females is $f_i x_i(t)$.Thus the number of female off-spring in the first age group at time $(t+1)$ is given by

$$x_1(t+1) = f_1 x_1(t) + f_2 x_2(t) + f_3 x_3(t) + \ldots + f_n x_n(t)$$

Now let $p_i,(i=1,2,...,n-1;0<p_i\le 1)$ be the proportion of females of the i-th age group at time t, who are surviving to become females of the (i+1)-th age group at time $(t+1)$. Now we consider the rate of harvesting $h_1,h_2,....,h_{n-1}$ of the population $x_2(t),x_3(t),...,x_n(t)$ respectively.

Therefore

$$x_{i+1} = p_i\, x_i(t) - h_i\, x_{i+1}(t) \; ; \; i = 1, 2, 3, \ldots n-1$$

Hence the model can be written as a system of difference equation

$$x_1(t+1) = f_1 x_1(t) + f_2 x_2(t) + f_3 x_3(t) + \ldots + f_n x_n(t)$$
$$x_2(t+1) = p_1 x_1(t) - h_1 x_2(t)$$
$$x_3(t+1) = p_2 x_2(t) - h_2 x_3(t)$$
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$$x_n(t+1) = p_{n-1} x_{n-1}(t) - h_{n-1} x_n(t)$$

Malthusian growth model:

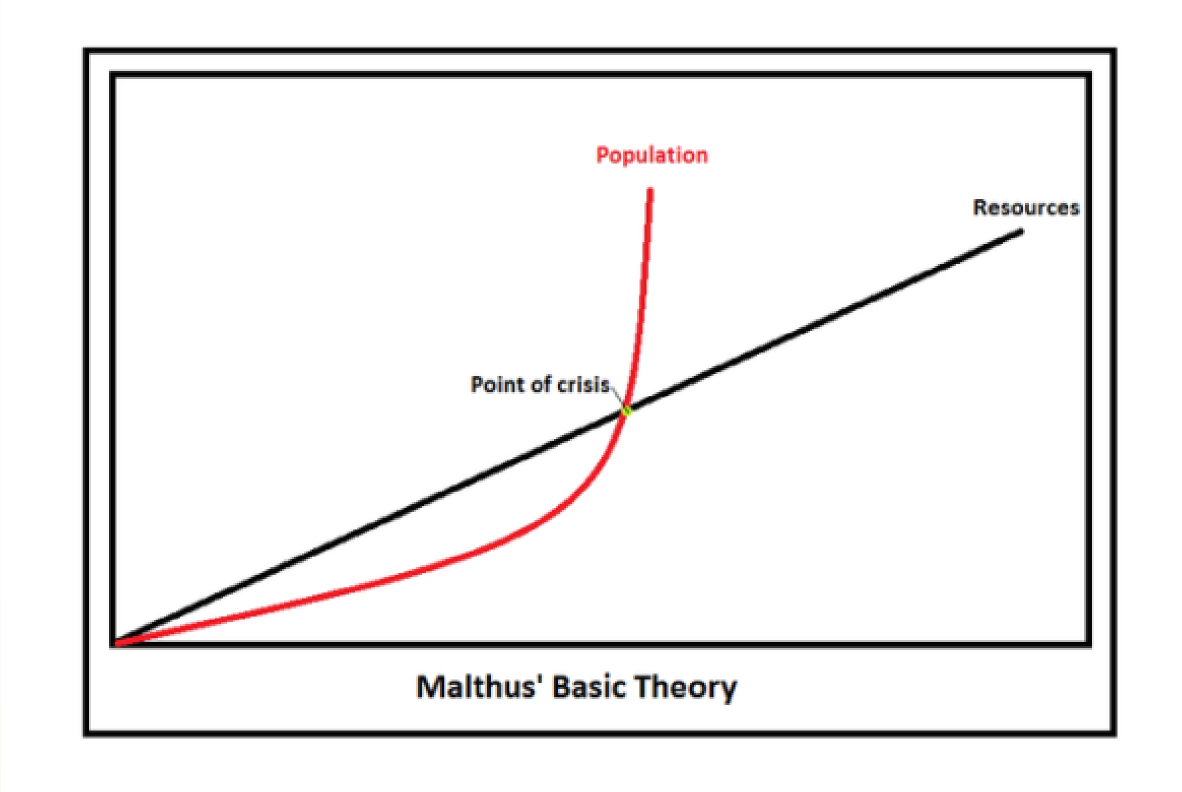
- A **Malthusian growth model**, sometimes called a **simple exponential** growth model, is essentially exponential growth based on a constant rate. The model is named after Thomas Robert Malthus, who wrote *An Essay on the Principle of Population* (1798), one of the earliest and most influential books on population.

Malthusian models have the following form:

$$\frac{dx(t)}{dt} = rx(t)$$

where $r\,(>0)$ is a constant called the intrinsic growth rate of the population.

Malthus Graph:

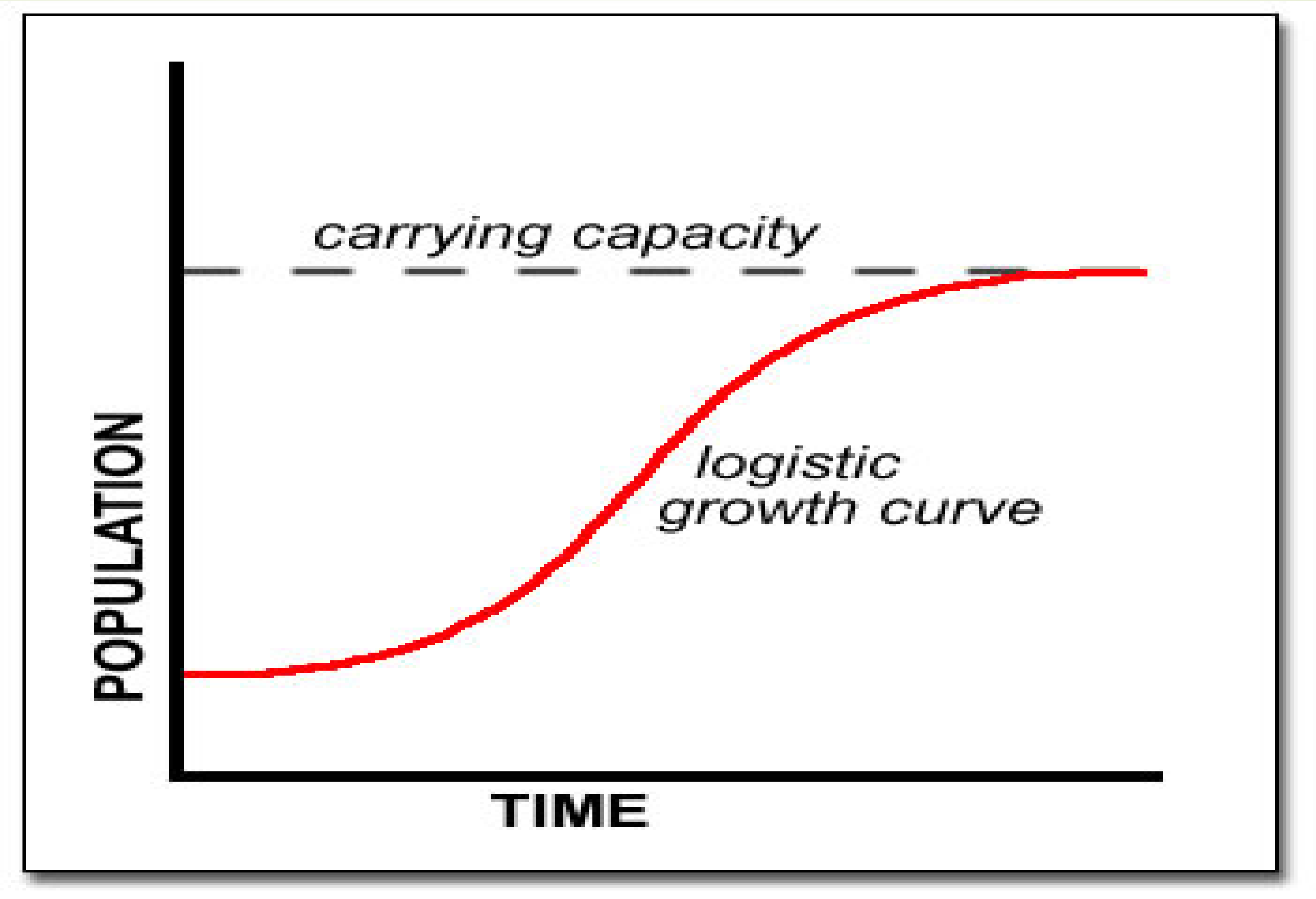


Logistic law of growth:

- $\frac{dx}{dt} = rx(1 - \frac{x}{k})$

- where r (> 0) is a constant called the intrinsic growth rate of the population.

- where k (> 0) is the carrying capacity of the habitat.


carrying capacity
logistic
growth curve
POPULATION
TIME

Gompertz law of growth:

- $\frac{dx}{dt} = r \log \frac{k}{x} x$

SIS epidemic model:

- **Susceptible-Infected-Susceptible**
- **You get sick, then recover, but without immunity**
- **E.g. the common cold.**

Diagram :

- Susceptible become infected at rate *a*
- Infected recover at rate *b*.
- $S \xrightarrow{a} I \xrightarrow{b} S$

SIS Equations:

- Becoming infected depends on contact
- between Susceptible and Infected (aSI)
- Recovery is at a constant rate, proportional to number of Infected (b).

$$\frac{ds}{dt} = bI - aSI$$

$$\frac{dI}{dt} = aSI - bI$$

Total population is constant:

- Add equations together
- $N = S + I$ (total population)
- $dN/dt = 0$, N is a constant.

$$\frac{dN}{dt} = \frac{ds}{dt} + \frac{dI}{dt} = bI - aSI + aSI - bI = 0$$

Books:

1. **Mathematical Biology: I. An Introduction: by:** by **James D. Murray**.
2. **Bio-Mathematics : by Pundir and Pundir.**
3. **Mathematical Modelling: by J. N. Kapur**

Thank You